

# Bessel Process and Conformal Quantum Mechanics

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**Abstract** Different aspects of the connection between the Bessel process and the conformal quantum mechanics (CQM) are discussed. The meaning of the possible generalizations of both models is investigated with respect to the other model, including self adjoint extension of the CQM. Some other generalizations such as the Bessel process in the wide sense and radial Ornstein-Uhlenbeck process are discussed with respect to the underlying conformal group structure.

**Keywords** Bessel Process · Conformal quantum mechanics · Self adjoint extension

## 1 Introduction

The Bessel process is one of the building blocks of stochastic processes because of its applications and also its simplicity and richness. The Bessel process describes the norm of the Brownian motion in arbitrary dimension [1, 2]. This process describes the movement of an arbitrary point on the real line in the Schramm-Loewner evolution [3]. It is also important in the probabilistic description of some financial markets specially Cox, Ingersoll and Ross (CIR) model [4]. Since the Bessel process is connected to the movement of a free random walk in arbitrary dimension it could be connected to the quantum mechanics of the free particle. The interpretation of quantum mechanics as the classical stochastic equation is a long story and was discussed in [5]. It is based on looking at the generator of the stochastic process as the Hamiltonian of the quantum mechanics. Although by now this interpretation is quite well known and was discussed also in many books our case, the Bessel process, has not been discussed as much as it deserves. The corresponding quantum mechanics is called conformal quantum mechanics and it has many applications in different areas of physics such as the Calogero model [6–11], conformally invariant quantum mechanical models [12–14], dynamics of quantum particles in the asymptotic near-horizon region of black-holes [15, 16] and dipole-bound anions as anisotropic conformal interaction [17]. Although this quantum

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mechanics was discussed from many different points of view, its connections and similarities with the Bessel process have not been addressed so far and that is our goal in this paper. From the stochastic processes point of view we have a continuous process which is also scale invariant. The same interpretation is true in the quantum mechanics part, but the story in the quantum mechanics side can be generalized by considering some special boundary conditions at the origin which can have stochastic interpretations. It is also possible to define other generalized Bessel processes which have interesting well-known quantum mechanical counterparts. We will address different aspects of these similarities and connections in this paper. The paper is organized as follows:

In the second section we address many different properties of the Bessel process, the starting point is our motivation from the norm of the Brownian motion but we will soon generalize the definition in many different directions. In the third section by starting from the quantum mechanics of the free particle we will extract conformal quantum mechanics as a quantum mechanics with conformal symmetry. Similar to the second section we will soon generalize the definition by allowing also a real and even an imaginary number of dimensions. Of course in every extension level we should be careful about the physical meanings and possible interpretations with respect to the original motivation. In this section we will also discuss the self adjoint extension of the corresponding quantum mechanics. In the fourth section we will discuss many connections and similarities between Bessel process and CQM, in particular we explain the possible meaning of the extension in each side with respect to the other. This comparison will also propose the necessity of studying generalized Bessel processes. In the fifth section we will investigate the most natural and simple generalizations of Bessel process including Bessel process with constant drift, Bessel process in the wide sense, CIR model and Morse process which are related to Coulomb potential, free particle conformal quantum mechanics, radial harmonic oscillator and Morse potential respectively. We explain how one can get the connections and also see the internal symmetries. Finally in the last section we will summarize our results.

## 2 Bessel Process

To define Bessel process we start with the primary motivation for definition of the process. It is just the radial part of  $\delta$ -dimensional Brownian motion, i.e.,  $R_t = \sqrt{B_1^2(t) + \dots + B_\delta^2(t)}$ . Using Ito's formula one can write the following stochastic equation for the Bessel process

$$dR_t = \frac{\delta - 1}{2R_t} dt + dB_t, \quad (2.1)$$

where  $B_t$  is the one dimensional Brownian motion. It is easy to see that the above process has scaling property, i.e. if  $R_t$  is a Bessel process with starting point  $x$  then the process  $c^{-1}R_{c^2t}$  is also Bessel process with starting point at  $x/c$  for positive  $c$ . In this level one can consider the above equation with the arbitrary real  $\delta$ . The Bessel process with positive  $\delta$  was studied extensively in the literature [1, 2]. Before establishing the properties of Bessel process it is worth mentioning some properties of Brownian motion in arbitrary dimension.

It is well known that Brownian motion in  $\delta \leq 2$  is recurrent and transient for  $\delta > 2$ . With transient we mean that if  $\delta > 2$  then almost surely  $\lim_{t \rightarrow \infty} |B_t| = +\infty$  and the process is recurrent if the set  $\{t : B_t \in U\}$  be unbounded for all the sets  $D$  in  $R^\delta$ ; in other words the process  $X_t$  said to be recurrent if  $P_x(T_y < \infty) = 1$  for all  $x, y \in D$  where  $T_y$  is the hitting time, the first time at which the process hits point  $y$ , and  $P_x(T_y < \infty)$  is the probability of

having finite hitting time of the point  $y$  for the process with starting point at  $x$ . Another interesting property of Brownian motion is related to the probability of meeting two Brownian paths with arbitrary starting points in finite time. Two Brownian paths will meet each other if  $\delta < 4$  otherwise they will not [18]. In other words almost surely the probability of intersection of two independent Brownian paths in  $\delta \geq 4$  is zero.

Let’s now summarize some of the properties of the Bessel process. Feller’s test, see appendix B, indicate that this process has a natural boundary at infinity and a boundary at origin which is natural if  $\delta \geq 2$ , regular if  $0 < \delta < 2$  and absorbing if  $\delta \leq 0$ . One can summarize also the following properties

- I* : for  $\delta > 2$  the Bessel process is transient,
- II* : for  $\delta \leq 2$  the Bessel process is recurrent,
- III* : for  $\delta \geq 2$  the origin is polar, it will not be touched by the process,
- IV* : for  $0 < \delta < 2$ , origin could be a killing point or reflecting point.

The same properties are true for the squared Bessel process defined by  $Z_t = R_t^2$  which satisfies the following equation

$$dZ_t = \delta(dt) + 2\sqrt{|Z_t|}dB_t. \tag{2.2}$$

Using Ito’s formula the generator of squared Bessel process is

$$\mathcal{L}f(x) := 2xf''(x) + \delta f'(x). \tag{2.3}$$

Then one may introduce the Green’s function  $G_\lambda(x, y)$  as the Laplace transform, with respect to time, of the transition density, the density of finding the process starting from  $x$  at  $y$ , if of the process

$$G_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y). \tag{2.4}$$

The Green’s function satisfies the following equation

$$\mathcal{L}G_\lambda(x, y) - \lambda G_\lambda(x, y) = 0. \tag{2.5}$$

The solution may be factorized as follows

$$G_\lambda(x, y) = \begin{cases} w_\lambda^{-1} \psi_\lambda(x) \phi_\lambda(y) & \text{if } x \leq y, \\ w_\lambda^{-1} \psi_\lambda(y) \phi_\lambda(x) & \text{if } x \geq y, \end{cases} \tag{2.6}$$

where  $w_\lambda$  is the Wronskian  $w_\lambda := \psi'_\lambda(x)\phi_\lambda(x) - \psi_\lambda(x)\phi'_\lambda(x)$ . Since in the case of  $0 < \delta < 2$  the process is touching zero we need to specify the boundary at the origin is a killing<sup>1</sup> boundary condition or reflecting boundary condition. Using the above boundary conditions one could find the following solutions

$$\psi_\lambda(x) = \begin{cases} x^{-\frac{\nu}{2}} I_\nu(\sqrt{2\lambda x}) & \text{if } \delta \geq 2 \text{ or } 0 < \delta < 2 \text{ with reflecting origin,} \\ x^{-\frac{\nu}{2}} I_{-\nu}(\sqrt{2\lambda x}) & \text{if } \delta \leq 0 \text{ or } 0 < \delta < 2 \text{ with killing origin,} \end{cases} \tag{2.7}$$

<sup>1</sup>In some literatures the word absorbing was used.

and

$$\phi_\lambda(x) = x^{-\frac{\nu}{2}} K_\nu(\sqrt{2\lambda x}), \tag{2.8}$$

where  $I_\nu$  and  $K_\nu$  denote the modified Bessel functions with index  $\nu = \frac{\delta-2}{2}$ . It is easy to see that  $w_\lambda = \frac{1}{2}$ . Using the above solutions one can write the transition density for all  $\delta > 0$  except the killing case as follows

$$p(t, x, y) = \frac{1}{2t} \left(\frac{x}{y}\right)^{-\frac{\nu}{2}} e^{-\frac{x+y}{2t}} I_\nu\left(\frac{\sqrt{xy}}{t}\right) \quad \text{if } x > 0, \tag{2.9}$$

$$p(t, x, y) = \frac{2}{(2t)^{1+\nu}\Gamma(1+\nu)} e^{-\frac{y}{2t}} \quad \text{if } x = 0. \tag{2.10}$$

The transition density for  $0 < \delta < 2$  with the killing origin is

$$p(t, x, y) = \frac{1}{t} \left(\frac{x}{y}\right)^{-\frac{\nu}{2}} e^{-\frac{x+y}{2t}} I_{-\nu}\left(\frac{\sqrt{xy}}{t}\right). \tag{2.11}$$

Similar results could be calculated for the Bessel process, in this case the generator has the following form

$$\mathcal{L}f(x) := \frac{1}{2}f''(x) + \frac{1}{2x}(\delta - 1)f'(x). \tag{2.12}$$

Since this process is just the square root of squared Bessel process one can get the transition densities in this case by just the transformations  $x \rightarrow x^2$  and  $y \rightarrow y^2$ . For example for the reflecting origin we will have

$$p(t, x, y) = \frac{1}{2t} y \left(\frac{x}{y}\right)^{-\nu} e^{-\frac{x^2+y^2}{2t}} I_\nu\left(\frac{xy}{t}\right). \tag{2.13}$$

One could get the same answer by the method which is more familiar for physicists and that is by using the Fokker-Planck equation which has the following form for the Bessel process

$$\partial_t p(t, x, y) = \frac{1}{2} \left( \partial_x^2 - \partial_x \frac{1+2\nu}{x} \right) p(t, x, y). \tag{2.14}$$

One can do the transformation  $p(t, x, y) = x^{\nu+\frac{1}{2}} Q(x, y, t)$  following by the Laplace transform and get the equation

$$-\frac{1}{2}\partial_x^2 Q_\lambda + \frac{\nu^2 - 1/4}{2x^2} Q_\lambda = -\lambda Q_\lambda, \tag{2.15}$$

where  $Q_\lambda$  is the Laplace transform of  $Q(x, y, t)$ . The above equation is just the modified Bessel equation with the modified Bessel functions as solutions. It is easy to see that the above eigenvalue problem by the change of variable  $S_\lambda = x^{-(\nu+1/2)} Q_\lambda$  is equivalent to  $(-\partial_x^2 - (\frac{1+2\nu}{x})\partial_x)S(x) = -\lambda S(x)$  which the operator is the same as the generator of Bessel process stated before.

So far we just addressed the Bessel process with natural boundary conditions in origin and infinity but it is also possible to investigate squared Bessel process with  $\delta > 0$  starting from the  $x \leq 0$  or squared Bessel process with  $\delta \leq 0$  and arbitrary starting point. These processes were studied in [19] and have the following properties: in the case  $\delta = 0$  the

process will reach zero and stays there. For a squared Bessel process with  $\delta \geq 0$  and starting point  $x \leq 0$  one could show that it behaves like  $-Z_{-x}^{-\delta}$  process, with starting point at  $-x$ , until it hits origin and after that it behaves like  $Z_0^\delta$ . Similar relations are valid for the process with  $\delta \leq 0$  and  $x > 0$  by just reversing the sign of dimension, starting point and the process. For example the process with negative dimension and negative starting point behaves as  $-Z_{-x}^{-\delta}$ . It is worth to mention that since for negative dimensions the squared Bessel process become negative the square root of it, which is Bessel process, will become purely imaginary and so one should be careful about extending the results to the Bessel process, however up to the time that the process is positive one could define the Bessel process as well as a real process. To complete the discussion we give the transition density of the squared Bessel process with negative dimension given in [19] for starting value  $x > 0$

$$\begin{aligned}
 p(t, x, y) &= k(x, y, \delta, t) e^{-\frac{x+|y|}{2t}} \int_0^\infty \frac{(1+w)^{-\delta}}{w^{-\delta/2}} \exp\left(\frac{-1}{2t}\left(xw + \frac{|y|}{w}\right)\right); \\
 k(x, y, \delta, t) &= \frac{-2^\delta}{\delta \Gamma^2\left[\frac{-\delta}{2}\right]} x^{1-\frac{\delta}{2}} |y|^{-(1+\frac{\delta}{2})} t^{\delta-1}.
 \end{aligned}
 \tag{2.16}$$

Another interesting process related to the Bessel process is the time reversed Bessel process. If  $R_t$  be a Bessel process with starting point on the positive real line with dimension smaller than two then the time reversed Bessel process, defined by  $(R_{(T_{x \rightarrow 0})-s}, s \leq T_{x \rightarrow 0})$ , has the same law with a Bessel process  $\hat{R}_s, s \leq \hat{L}_{0 \rightarrow x}$  starting from origin with  $\hat{\delta} \equiv 4 - \delta$  and  $\hat{L}_{0 \rightarrow x} \equiv \{\sup t | \hat{R}_t = x\}$ . In the above notation  $T_{x \rightarrow y} \equiv \inf\{t | R_t = y\}$  is the first time that the process starting at  $x$  hits the point  $y$ . To make it more clear consider the ensemble of all Bessel paths that ended up when they hit the origin in the first time, then consider all these paths in the reverse time, the laws of these two processes are the same. The other way to say the above statement is as follows: For every bounded function  $F$  one can write

$$E_0^\delta[F(R_{(T_{x \rightarrow 0})-s}, s \leq T_{x \rightarrow 0})] = E_0^{4-\delta}[F(\hat{R}_s, s \leq \hat{L}_{0 \rightarrow x})],
 \tag{2.17}$$

where  $F$  is a function on the set of realizations of the process. The above equality relates Bessel process with dimension  $\delta$  to Bessel process with dimension  $\delta' = 4 - \delta$ . One can write this duality by defining  $\kappa = \frac{4}{1-\delta}$  as  $\frac{1}{\kappa} + \frac{1}{\kappa'} = \frac{1}{2}$ . Using the equality one could easily see that  $n = \sin(\frac{4\pi}{\kappa}) = \sin(\frac{4\pi}{\kappa'})$ . The above equality is valid for all real values of  $\delta$ . It is also possible to write the equality as  $\nu + \hat{\nu} = 0$ , then one could say that these two processes have the same  $Q(x, y, t)$  if we work in the positive range of dimensions of Bessel process. In fact we will see in section two that these two different Bessel processes are related to the same conformal quantum mechanics. It is also easy to see that  $\delta = 2$  is self dual and  $\delta = 4$  is critical which beyond that our reverse time process has negative dimension. The above equality is also useful to get the law of certain first hitting times of Bessel process [19].

### 3 Conformal Quantum Mechanics

In order to introduce conformal quantum mechanics we follow the same strategy that we had in the previous section, we start from the free particle in  $\delta$  dimensions. The Schrödinger equation for a free particle in radial coordinates has the following form

$$-\frac{1}{2r^{\delta-1}} \frac{\partial}{\partial r} \left( r^{\delta-1} \frac{\partial \varphi(r)}{\partial r} \right) = E \varphi(r),
 \tag{3.1}$$

or after differentiation one could write the following Bessel generator eigenvalue equation

$$-r \frac{\partial^2 \varphi(r)}{\partial r^2} - \frac{\delta - 1}{r} \frac{\partial \varphi(r)}{\partial r} = 2E\varphi(r). \tag{3.2}$$

As it is evident a similar equation was derived for the Bessel process in Laplace space in the previous section. By  $\varphi(r) = r^{-(\nu+1/2)} Q(r)$  and extending the range of  $\delta$  to all real values we will have the following Hamiltonian for conformal quantum mechanics

$$H = \frac{1}{2} \left( p^2 + \frac{\nu^2 - 1/4}{r^2} \right), \tag{3.3}$$

where  $p = i \frac{\partial}{\partial r}$  and  $[r, p] = i$  at the quantum level. The above quantum mechanics as a singular quantum mechanics was discussed extensively soon after the discovery of quantum mechanics [20, 21] and references therein. It is easy also to get the same quantum mechanics for two free particles moving in  $\delta$  dimensions with  $\frac{1}{r^2}$  interaction by just forgetting the momentum of the center of mass term and going to the radial part of the spherical coordinates. The conformal quantum mechanics has many interesting properties. Before investigating different solutions of the above system we study some symmetries of the system. The Hamiltonian at the level of one dimensional quantum mechanics with usual inner product for  $Q(r)$  is symmetric with respect to  $\nu \rightarrow -\nu$  but at the level of  $\varphi(r)$  is not symmetric with respect to the same transformation. The action corresponding to the Hamiltonian (3.3) is invariant under the following transformations

$$\begin{aligned} t' &= \frac{at + b}{ct + d}, \\ r'(t') &= \frac{r(t)}{ct + d} \quad \text{with } ad - bc = 1. \end{aligned} \tag{3.4}$$

The above transformations are the conformal transformations in 0 + 1 dimensions [12]. The basic transformations are time translation, dilation and special conformal transformation with the following Noether charges

$$t' = t + b, \quad H = H = \frac{1}{2} \left( p^2 + \frac{\nu^2 - 1/4}{r^2} \right); \tag{3.5}$$

$$t' = \alpha^2 t, \quad D = tH - \frac{1}{4}(rp + pr); \tag{3.6}$$

$$t' = \frac{t}{ct + 1}, \quad K = t^2 H - \frac{1}{2}t(rp + pr) + \frac{1}{2}r^2. \tag{3.7}$$

The above generators verify the algebra of the conformal group  $SO(1, 2)$  which is

$$[H, D] = iH, \quad [D, K] = iK, \quad [H, K] = 2iD. \tag{3.8}$$

Using the new definitions  $L_0 = \frac{1}{2}(H + K)$  and  $L_{\pm 1} = \frac{1}{2}(H - K \pm 2iD)$  one can write the algebra in the more familiar form

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, \quad [L_{+1}, L_{-1}] = 2L_0. \tag{3.9}$$

Explicit dependence of  $K$  and  $D$  on  $t$  will guarantee their conservations. The important thing to mention is that the argument is not considering the most general case because we

already knew that the action is invariant up to a total derivative which could be non zero in the presence of the boundary in the origin. We will discuss both conformal invariant and anomalous case. This will be more clear when we discuss the self adjoint extension of the quantum mechanics. Another important thing to mention is although it seems that we found three conservation laws for our two-dimensional phase system, it is not difficult to check that they are in fact related by the following relation

$$HK + KH - 2D^2 = \frac{\nu^2 - \frac{1}{4}}{2} - \frac{3}{8}, \tag{3.10}$$

which is also the Casimir operator of the group. In order to present different aspects of the above quantum mechanics we introduce another variable

$$g = \nu^2 - \frac{1}{4}. \tag{3.11}$$

Our quantum mechanics has different properties with respect to the value of  $g$ , some of which we will summarize in the following. For an arbitrary value of  $g$  it is easy to show that if one could find one bound state with energy  $E$  then scaling the position by an arbitrary factor  $\alpha$  it is easy to construct a new solution with energy  $\alpha^2 E$ , which means that if there exist any bound states then there is a bound state for every negative energy. This is a direct consequence of the scaling property of this model. The same story is true for the positive energy solutions of the model which means that we have just planar waves with all the possible positive energies which was also used to obtain (2.15). In the previous section we considered  $\lim_{r \rightarrow 0} \varphi(r) = 0$  for the killing boundary condition and for the reflecting case we had  $\lim_{r \rightarrow 0} \frac{\partial \varphi(r)}{\partial r} = 0$ . In the level of conformally invariant quantum mechanics we will ignore the negative energy solutions which means that our Hilbert space is made by wave functions corresponding to continuous positive energy solutions. The corresponding boundary condition also will be discussed in the end of this section.

The Hamiltonian could have other solutions dependent on the value of the coupling  $g$ . Firstly we should mention that the Laplacian operator that was the starting point is self adjoint if we consider it in the whole space. But one could extract other solutions by removing the origin from the domain and considering the self adjoint extension of the operator in the new domain. Of course existence of the extension is dependent on the value of  $g$  or in other words to the corresponding dimension. All the necessary aspects of theory of self adjoint extension were discussed in the Appendix A. In the case of our quantum mechanics the possible extensions were discussed in [22–24].

For  $g \geq \frac{3}{4}$  the Hamiltonian is self adjoint with removed origin now and has a scattering sector with the following solutions

$$\varphi(r) = (\sqrt{2Er})^{\frac{1}{2}} J_\nu(\sqrt{2Er}) \quad \text{or} \quad (\sqrt{2Er})^{\frac{1}{2}} Y_\nu(\sqrt{2Er}), \tag{3.12}$$

where  $J$  and  $Y$  are the Bessel functions. For  $\nu \geq 1$  just  $J$  can be considered because in this case  $Y$  at the origin is infinity and for  $\nu \leq -1$  just  $Y$  has desired property. For  $\nu \geq 1$  which is equivalent to  $\delta \geq 4$  one could argue that it is not possible to have a bound state solution because always we can just consider one of the solutions of the Bessel equation. The boundary condition in this case is  $\lim_{r \rightarrow 0} \varphi(r) = 0$  and can not be extended.

For  $g < \frac{3}{4}$  since the Hamiltonian is not self adjoint one can find a required self adjoint extension. It is better to distinguish between the domain  $-\frac{1}{4} \leq g < \frac{3}{4}$  and  $g < -\frac{1}{4}$ . First we discuss the domain  $-\frac{1}{4} \leq g < \frac{3}{4}$  which is equivalent to  $-1 < \nu < 1$  or  $0 < \delta < 4$ . In this

range the Hamiltonian requires a self adjoint extension with the self adjoint parameter  $z$  which is responsible to map two deficiency subspaces by the unitary map  $e^{iz}$ . The important point to mention is that although the Hamiltonian in the domain  $D(H) \equiv \{\varphi(0) = \varphi'(0) = 0\}$  is not self adjoint, it is still Hermitian.

If we consider  $H^\dagger$  as the adjoint of  $H$ , with the same differential representation as  $H$ , then from the Von Neumann's theory of deficiency indices we know that the deficiency subspaces  $K_\pm$  are made by the square integrable solutions of the equation  $H^*\phi_\pm = \pm i\phi_\pm$  in the desired domain. In our case the solutions are

$$\phi_+(r) = r^{\frac{1}{2}} H_\nu^1(r e^{i\frac{\pi}{4}}), \tag{3.13}$$

$$\phi_-(r) = r^{\frac{1}{2}} H_\nu^1(r e^{-i\frac{\pi}{4}}), \tag{3.14}$$

where  $H_\nu^1$  is the Henkel function and both of  $\phi_\pm(r)$  are square integrable in the half line. Using the above solutions one could argue that the Hamiltonian is self adjoint in the domain  $D_z(H) = D(H) \oplus \{u(\phi_+(r) + e^{iz}\phi_-(r))\}$  where  $u$  is an arbitrary complex number. To find the valid boundary condition we need  $\psi$  to be in the self adjoint domain. Consider  $\Phi = \phi_+(r) + e^{iz}\phi_-(r)$  then  $\psi$  is in the self adjoint domain if  $\langle \Phi | H \psi \rangle = \langle H \Phi | \psi \rangle$  or

$$\lim_{r \rightarrow 0} \left[ \Phi^* \frac{d\psi}{dr} - \psi \frac{d\Phi^*}{dr} \right] = 0. \tag{3.15}$$

To check this we need the behavior of  $\Phi$  close to the origin

$$\Phi(r) \rightarrow \frac{i}{\sin(\pi\nu)} [Ar^{\nu+\frac{1}{2}} + Br^{-\nu+\frac{1}{2}}], \tag{3.16}$$

where

$$A = \frac{e^{-i\frac{3\pi\nu}{4}} - e^{i(z+\frac{3\pi\nu}{4})}}{2^\nu \Gamma(1+\nu)}, \tag{3.17}$$

$$B = \frac{e^{i(z+\frac{\pi\nu}{4})} - e^{-i\frac{\pi\nu}{4}}}{2^{-\nu} \Gamma(1-\nu)},$$

for  $\nu \neq 0$ . Using the above relations one can write the equation (3.16) for the boundary condition as follows

$$(Ar^{\nu+1/2} + Br^{-\nu+1/2}) \frac{d\psi}{dr} - \left( A \left( \nu + \frac{1}{2} \right) r^{\nu-1/2} + B \left( \frac{1}{2} - \nu \right) r^{-\nu-1/2} \right) \psi \rightarrow 0. \tag{3.18}$$

The equation for  $\nu = \frac{1}{2}$  is like  $B \frac{d\psi}{dr} |_0 = A\psi(0)$ . Actually  $\nu = \frac{1}{2}$  is very interesting because firstly it corresponds to  $\delta = 3$  and also because it describes the possible boundary conditions for the free particle in the half line. The simplicity of the results helps us to investigate the possible meanings of the above self adjoint extension with respect to the Brownian motion and the Bessel process which we will discuss in more detail in the next section. The next important case which deserves separate calculations is  $\nu = 0$ . In this case the function  $\Phi$  has different properties near the origin. Using  $\phi_+(r) = \phi_-^*(r)$  and

$$\begin{aligned} \Phi(r) &\rightarrow \frac{2i}{\pi} r^{\frac{1}{2}} \ln(r) + \left[ \frac{1}{2} + \frac{2i}{\pi} (\gamma - \ln 2) \right] r^{\frac{1}{2}} + e^{iz} cc \\ &= (A + A^* e^{iz}) r^{\frac{1}{2}} \ln(r) + (B + B^* e^{iz}) r^{\frac{1}{2}}, \end{aligned} \tag{3.19}$$



where  $\gamma$  is the Euler constant and  $cc$  is the conjugate of the first term. One can find the following boundary condition

$$Ar^{\frac{1}{2}} \ln(r) \frac{d\psi}{dr} - \frac{1}{\sqrt{r}} \left[ \mathbf{A} \left( 1 + \frac{\ln r}{2} + 1 \right) + \frac{\mathbf{B}}{2} \right] \psi \rightarrow 0, \tag{3.20}$$

where

$$\mathbf{A} = A + A^* e^{iz}, \quad \mathbf{B} = B + B^* e^{iz}. \tag{3.21}$$

One can extend the results also for the case with  $g < -\frac{1}{4}$  which is related to pure complex  $\nu$ . In this case we have still a one parameter self adjoint extension and all of the results are still valid if we put imaginary  $\nu$  in to the formulas, in particular in to the (3.18). The important point to mention is in fact that by introducing the self adjoint parameter we are classifying all the possible physical boundary conditions of our system. In our case we should mention that after the self adjoint extension our Hamiltonian is not necessarily scale invariant and so could have a negative ground state which is in fact the case here. Of course the ground state depends on the self adjoint extension parameter. The energy and wave function for our Hamiltonian were discussed before in [23] and references therein, and the results are as follows.

The Hamiltonian does not have any bound state for  $g \geq \frac{3}{4}$  but admits one bound state in the range  $-\frac{1}{4} \leq g < \frac{3}{4}$ . The wave function of the bound state up to the normalization constant has the following form

$$\psi(r) \sim r^{\frac{1}{2}} K_\nu(\sqrt{-2E_\nu}r), \tag{3.22}$$

with energies

$$E_{\nu \neq 0} = - \left( \frac{\sin(\frac{z}{2} + \frac{3\pi\nu}{4})}{\sin(\frac{z}{2} + \frac{\pi\nu}{4})} \right)^{\frac{1}{\nu}}, \tag{3.23}$$

$$E_{\nu=0} = - \exp \frac{\pi}{2} \cot \left( \frac{z}{2} \right). \tag{3.24}$$

For the case  $g < -\frac{1}{4}$  we have infinite discrete bound states [20] with the same wavefunction as above but with imaginary  $\nu$ . To get the energies we just need to put the wave function in the corresponding boundary condition which yields

$$E = -2 \left( \frac{A}{B} \right)^\nu \exp\{2 \arg \Gamma(1 + \nu) - 2n\pi\}, \quad \text{for } n \in \mathbf{Z}. \tag{3.25}$$

It is worth to emphasize again that since after the self adjoint extension the action is not scale invariant we do not need to worry about having a bound state because it does not enforce any other bound state by the scaling argument.

One could summarize this section as follows: we introduced conformal quantum mechanics as the quantum mechanics of the free particle in  $\delta$  dimensions. Constraining the domain of the quantum mechanics to the space without origin forces new boundary conditions to the system. By these boundary conditions for some values of  $g$  we need to find a self adjoint extension of the Hamiltonian. The method of extension helps to find all the relevant boundary conditions and so the physics of the model is related drastically to the self adjoint extension parameter. The extended Hamiltonian which is not scale invariant admits discrete

bound states. Since after the self adjoint extension our conformal symmetry has some anomalies, may be the expression conformal quantum mechanics is a bit confusing but because of our first motivation we will keep using it. Different aspects of this anomaly were discussed extensively in [17, 25–27] from the re-normalization group point of view. The above results could be also explained from the framework of two free particles moving in  $\delta$  dimensions. The self adjoint extension is just giving boundary conditions corresponding to the time that two particles meet each other. For example, bound states of the above system could be seen as the bound states of two particles which behave like one particle. In the next section we give some hints on how one can translate the above results in the Brownian motion language.

#### 4 Bessel Process and CQM

In this section we are interested in discussing different features of the similarity between Brownian motion in  $\delta$  dimension and free particles from a quantum mechanics point of view and also similarities between Bessel process and CQM.

The first remark is for the Bessel process with positive dimension. We had origin and infinity as the natural boundary conditions which is also the case for our CQM. In addition before extending the Hamiltonian we have a scale invariant system which is the case also in the Bessel process. Just as we remarked in the previous section, keeping the scaling symmetry of the CQM gives us an unbounded continuous spectrum which is also true if we try to find the transition density of the Bessel process. To make more clear the connection between two models one can use the following relation between the transition density of the Bessel process and the path integral of CQM

$$p(x, y, t) = \int_{q(t_0)=x}^{q(t)=y} [dq(\tau)] \exp[-S(q(\tau))], \quad (4.1)$$

$$S(q) = \int \frac{1}{2} \left( \dot{q} + \frac{1-\delta}{2q} \right)^2 dt. \quad (4.2)$$

It is not difficult to see that by a transformation  $p(t, x, y) = x^{\nu+\frac{1}{2}} Q(x, y, t)$  one can go from the above action to the familiar conformal action that we discussed in the previous section.

By the above discussion it seems that the self adjoint extension is something beyond our normal understanding of the Bessel process. It is not difficult to see that it is related to the different possible boundary conditions for the Bessel process at the origin. Different self adjoint extensions are related to different boundary conditions or to different measuring of the paths of the Bessel process. To make it more rigorous, by Bessel process we mean some kind of generalized Bessel process which is related to the diffusion of Brownian motion in the space without the origin. From the two Brownian motions point of view naively one could argue that since we have a self adjoint extension just for  $\delta < 4$  it is just a hint that two Brownian paths could meet in just  $\delta < 4$  dimensions. Of course we are aware that these two statements are not equal, but they are related because in the case of two Brownian paths one could think about moving one Brownian motion in the space which remains after removing from the original space the region filled by another Brownian motion and the boundary condition on the fixed Brownian motion is the same as the one that we discussed before. In fact the interaction of the paths is a point like interaction and as we will explain soon this interaction plays no role in dimension greater than three in quantum mechanics, neither for attractive nor for repulsive interactions.

To understand what could be the meaning of self adjoint extension in the case of Brownian motion, it is worth to understand the situation first in the case of quantum mechanics. One way to get a motivation for the self adjoint extension in the pointed space  $\mathbf{R}^\delta$  is by considering the delta function potential at the origin. Of course this will work just for the integer  $\delta$ . The cases  $\delta = 1, 2, 3$  are studied with more details for two reasons, firstly they have more applications and secondly the  $\delta \geq 4$  cases are trivial in some senses that we will discuss soon. The equality of a Hamiltonian with a delta function potential with a free Hamiltonian on a space with one point deleted plus a boundary condition were discussed by Jakiew [28]. The equality is in the level of re-normalized delta function which gives the same scattering data and bound states as produced by the self adjoint extension. One could re-normalize the delta function with different functions including sphere delta function, square well and lattice regularization. For example in the sphere delta function case, to regularize  $V(\mathbf{r}) = v\delta(\mathbf{r})$  we can use

$$V(r) = \frac{c_d v}{2\pi R^{d-1}} \delta(r - R), \quad \text{with } c_2 = 1, \quad c_3 = \frac{1}{2}; \tag{4.3}$$

for dimensions  $d = 2, 3$ . The corresponding limit is  $R \rightarrow 0$ . In the case of the square well we have  $V(\mathbf{r}) = -D\Theta(R - r)$ , where  $\Theta$  is the Heaviside function,  $D$  is a constant and the limit is the same. Since these regularization procedures were discussed extensively, we ignore the details but we explain the strategy, for more details see [28]. Using the above regularized delta potentials firstly we should solve the problem for the regularized potential by considering the continuous wave function at  $r = R$ . This matching leads to equations between different parameters including  $v, R$  and energy. Then by defining the re-normalized coupling  $\tilde{v}$  as a function of  $R$  and  $v$ , one could derive the energy and wave function of the system exactly as we found in the previous section. For example for the three dimensional case we have by definition  $\frac{1}{\tilde{v}} = \frac{1}{v} + \frac{1}{2\pi R}$  and after matching the wave functions we will have the following equations for the bound state energy  $E_b$  and the phase shift of the  $s$ -wave scattering sector

$$\sqrt{2E_b} = \frac{2\pi}{\tilde{v}}, \tag{4.4}$$

$$\tan(\delta_0) = -\frac{\tilde{v}k}{2\pi}. \tag{4.5}$$

Similar results for the two dimensional delta function were given in [28].

This regularization procedure will not work for dimensions  $d \geq 4$  because it is not possible to absorb all of the divergent terms in this case. Although it seems that the re-normalized attractive delta function in two dimensions has the same physics as the free particle quantum mechanics in the plane without origin, this matching is not complete for some reasons. For explanation we need to make some remarks on the general properties of the delta function potential in arbitrary dimensions. Firstly the repulsive delta function is trivial in dimensions higher than one which means that the phase shift scattering is zero for this case and so the scattering matrix is equal to one [29]. For the attractive case as we remarked before it is not possible to define a zero range potential in more than three dimensions possessing bound states with finite energy. For simplicity we focus on the three dimensional case and compare with more detail the delta function potential and the self adjoint extension counterpart. Let's take the subset of the possible boundary conditions with the following property for each real  $\eta$ ,

$$\psi(0) = \frac{-\eta}{2\pi} \frac{d\psi}{dr} \Big|_{r=0}. \tag{4.6}$$

For positive energy  $E = k^2/2$  the solution is as follows

$$\psi(r) = \frac{1}{r}(\sin(kr) + \tan(\delta_0) \cos(kr)), \tag{4.7}$$

where  $\delta_0$  is the phase shift corresponding to the  $s$ -wave. It is not difficult to see that for the attractive delta function using (4.6) and (4.7) one could get  $\eta = \tilde{\nu}$ . This matching can not be done for the repulsive case because we already know that the repulsive delta function is trivial, but the self adjoint extension with  $\eta < 0$  still has a phase shift and so it is not trivial. At the level of the attractive case one may still interpret the physics of the self adjoint extension for the integer  $\delta$  as the stochastic process in the presence of the regularized delta function. However, writing a stochastic equation for the radial Brownian motion with one removed point is not obvious and needs more investigation. Of course the definition of a stochastic process for the generic case with arbitrary  $\delta$  or  $\nu$  is more difficult. Although the boundary condition for  $\delta = 3$  or  $\nu = \frac{1}{2}$ , which is equivalent to the half line free quantum particle, is very simple, it is enlightening. One could see the boundary condition (4.6) from the stochastic process point of view at one dimension as follows:  $\eta = 0$  is the Dirichlet boundary condition which is an absorbing boundary condition. The particle will be absorbed by the origin after hitting that. This also corresponds to the conventional free Hamiltonian with scale invariance. From the two particle point of view it is like absorption of one particle by the other one when they touch each other. The Dirichlet boundary case is just reminiscent of the equality of the 3 dimensional Bessel process with the Brownian motion on the half line with absorbing boundary condition.

The other extreme case is the Neumann boundary condition  $\eta \rightarrow -\infty$  which is a reflecting boundary condition at the origin for the Brownian motion. Other negative values of  $\eta$  correspond to a mixing of Dirichlet and Neumann boundary conditions. This correspondence was discussed in detail in [30–32] and the Green function has the following form for arbitrary value of the self adjoint extension

$$G_\eta(x, y, t) = G_F(x - y, t) + G_F(x + y, t) + \frac{4\pi}{\eta} \int_0^\infty dw e^{\frac{2\pi}{\eta}w} G_F(x + y + w), \quad \eta \leq 0; \tag{4.8}$$

$$G_\eta(x, y, t) = G_F(x - y, t) + G_F(x + y, t) - \frac{4\pi}{\eta} \int_0^\infty dw e^{-\frac{2\pi}{\eta}w} G_F(x + y - w) + \frac{4\pi}{\eta} e^{i\frac{2\pi^2 t}{\eta^2}} e^{-\frac{2\pi}{\eta}(x+y)}, \quad \eta \geq 0; \tag{4.9}$$

$$G_F(x - y, t) = \frac{1}{\sqrt{2\pi it}} e^{i(x-y)^2/2t}. \tag{4.10}$$

For the special cases, Dirichlet and Neumann the results are as follows

$$G_{\eta=0}(x, y, t) = G_F(x - y, t) + G_F(x + y, t); \tag{4.11}$$

$$G_{\eta \rightarrow -\infty}(x, y, t) = G_F(x - y, t) - G_F(x + y, t). \tag{4.12}$$

One can use the above equations to get the Green’s function of the Brownian motion in three dimensions when the origin is removed. Of course one could get the Green’s function for the general case by using an orthogonal eigenvalue expansion for the self adjoint operator. The

Green’s function with respect to the solutions of the Hamiltonian has the following form

$$G_\eta(x, y, t) = \int_0^\infty dk e^{-iE_k t} \varphi(y) \varphi^*(x). \tag{4.13}$$

In the case of  $g < \frac{3}{4}$  since we have also a bound state we need to add new terms coming from the discrete contribution of this states by adding  $\sum_b (e^{iE_b t} \psi(r) \psi^*(r))$  to the Green’s function, where  $E_b$  is the bound state energy. The important conclusion is that one could also consider the above Green’s function as sort of a generalized Bessel process in  $\delta$  dimensions. In fact one can get the transition density for the above processes by just Wick rotation. The above generalization of Bessel process has not been appeared in the mathematical literature. From the stochastic process point of view one can derive the above solutions by considering the local time of the process [1]. The definition of the local time of the path  $\omega$  at the point  $a$  is as follows

$$t_l(a) := \frac{1}{2} \lim_{\epsilon \rightarrow 0} \int_0^T \mathbf{1}_{x+\epsilon}(B_s) ds, \tag{4.14}$$

where  $\mathbf{1}_{x+\epsilon}(B_s)$  is the indicator for the time that the process is in the interval  $[0, x + \epsilon]$ . One could naively write the above equation as an integral over a delta function as  $t_l(\omega, 0) = \int_0^T \delta(x(t)) dt$ . Using the above equation the extended transition density for  $\delta = 3$  is just the expectation value of  $\exp(\frac{-2\pi}{\eta} t_l)$ . The corresponding stochastic process for the free particle in the half line is called elastic Brownian motion [1]. To the best of our knowledge the problem has not been discussed by the mathematicians for the generic case. For the Bessel process that we discussed in the second section one just needs to consider  $B = 0$  in the (3.18). In this case we will recover conformal symmetry for our process again. It will be also interesting if one could get the results for the Bessel process with negative dimension by using the quantum mechanics of a free particle, in particular (2.16).

### 5 Generalization of Bessel Process and CQM

In this section we want to present some possible generalizations of the Bessel process with the well known quantum mechanical counterparts such as, Bessel process with constant drift, Bessel process in the wide sense, Cox, Ingersoll and Ross (CIR) model and Morse process which are related to a Coulomb potential, conformal quantum mechanics, radial harmonic oscillator and Morse potential respectively. Of course not all of the above processes admit conformal symmetry; they are reasonable perturbations of the conformal invariant case.

#### 5.1 Bessel Process with Constant Drift and Coulomb Potential

The definition of the Bessel process with constant drift is given by the following stochastic process

$$dR_t^\mu = \left( \frac{\delta - 1}{2R_t} + \mu \right) dt + dB_t. \tag{5.1}$$

From the Feller classification the boundary at the origin has the same classification as for the standard Bessel process, the infinity is a natural boundary condition which is attracting

if  $\mu$  is positive and non-attracting for a negative drift. The generator of the process is

$$\mathcal{L}f(x) := \frac{1}{2}f''(x) + \left(\frac{1}{2x}(\delta - 1) + \mu\right)f'(x). \quad (5.2)$$

One could map the solution of (2.5) to the Schrödinger equation of the Coulomb potential. To do so we need the Liouville transformation

$$R_\lambda(r) = 2^{\frac{1}{4}} \left(\frac{r}{\sqrt{2}}\right)^{\frac{\delta-1}{2}} e^{\frac{\mu r}{\sqrt{2}}} G_\lambda\left(\frac{r}{\sqrt{2}}\right), \quad (5.3)$$

where  $r = \sqrt{2}x$ . Using the above transformation one could write (5.2) as

$$\frac{d^2 R_\lambda}{dr^2} + \left(\frac{c}{r} - \frac{l(l+1)}{r^2}\right) R_\lambda = -E R_\lambda, \quad (5.4)$$

where  $E = -\lambda + \frac{\mu^2}{2}$ ,  $l = \frac{\delta-1}{2}$  and  $c = -\mu\frac{\delta-1}{\sqrt{2}}$ . For integer  $\delta$  the equation is related to the energy levels of a Hydrogen atom. The different aspects of (5.4) were discussed in [33] including the transition density for the different values of  $\mu$  and  $\delta$ , here we just mention some properties. The Bessel process with constant drift admits a discrete spectrum for all negative values of  $\mu$  which is expectable for those people who are familiar with the Hydrogen atom. For  $0 < \delta < 1$  the process admits just one trivial bound state but for  $\delta > 1$  it has infinitely many bound states. The process also admits discrete spectrum for the positive values of  $\mu$  for dimensions in the range  $0 < \delta < 1$ . In this case  $\delta = 1$  is the critical dimension for admitting discrete energies or not.

## 5.2 Bessel Process in the Wide Sense

Another very important generalization of the Bessel process is by considering a Brownian motion in  $\delta$  dimensions with a drift of magnitude  $\mu \geq 0$  and then taking the radial part as a time homogeneous diffusion in  $[0, \infty)$ . It was shown in [35] that the generator of this process is

$$\mathcal{L}f(r) := \frac{1}{2}f''(r) + \left(\frac{1}{2r}(\delta - 1) + h_v^{-1}(\mu r)\frac{dh_v(\mu r)}{dr}\right)f'(r), \quad (5.5)$$

where

$$h_v(x) = \left(\frac{2}{x}\right)^v \Gamma(1+v)I_v(x), \quad v = \frac{\delta}{2} - 1. \quad (5.6)$$

The transition density of this process is related to the transition density of the Bessel process  $p_\delta(t, x, y)$  as follows

$$p_{\delta,\mu}(t, x, y) = e^{-\mu^2 t/2} h_v^{-1}(\mu x) p_\delta(t, x, y) h_v(\mu y). \quad (5.7)$$

The above process has an interesting time inversion property. It was shown in [34], see also [36] that if  $X_t$  with  $t \geq 0$  is a diffusion process and  $tX_{\frac{1}{t}}$  with  $t > 0$  is homogeneous and conservative, no killing in the interior domain, then both processes are necessarily Bessel processes in the wide sense up to a time scaled re-parametrization. It is easy to generalize the above statement to the processes with time inversion property of degree  $\alpha$  by demanding a Markov process with homogenous  $t^\alpha X_{\frac{1}{t}}$ . They can be obtained by just considering Bessel

processes in the wide sense with appropriate power. The Hamiltonian corresponding to this process after transformation  $p_{\delta,\mu}(t, x, y) = r^{v+1/2}h_\nu(\mu r)Q_\nu(\mu r)$  is as follows

$$\begin{aligned}
 H &= \frac{1}{2}(p^2 + V(r)), \\
 V(r) &= \frac{v^2 - 1/4}{r^2} + \mu^2,
 \end{aligned}
 \tag{5.8}$$

where we used the equality  $\mu^2 = \frac{3}{4}\mu^2 + \frac{\mu}{4r} \frac{(\delta I_{1+\nu}(\mu r) + \mu r I_{2+\nu}(\mu r))}{I_\nu(\mu r)}$ . Since the third term does not have any singularity it means that the general aspects of the above quantum mechanics are the same as the Bessel process, in particular it will have a self adjoint extension for the range  $\nu < 1$ . This equality means that basically the above two distributions have similar spectrum and also it is a very simple proof for (5.7). One could use all the previous solutions for the self adjoint Hamiltonian to get the solutions in this case. The consistency of the inner product is coming from the Doob’s  $h$ -transform. For the case of Brownian motion with negative drift  $-\mu$  one just needs to replace the modified Bessel function  $I_\nu(\mu r)$  with  $K_\nu(\mu r)$ . The same as the Bessel process case here also one can generalize the Bessel process in the wide sense by considering all the possible self adjoint solutions of the Hamiltonian (5.8) and then using the Doob’s  $h$ -transform. It will be really interesting to study this generalized cases with respect to the time inversion.

### 5.3 CIR Model and Radial Harmonic Oscillator

The Cox, Ingersoll and Ross (CIR) model [4] or radial Ornstein- Uhlenbeck process is widely used for interest rate framework such as some stochastic volatility models [37]. The definition of the CIR family of diffusions is by the following equation

$$dN_t = (a - bN_t)dt + c\sqrt{|N_t|}dB_t,
 \tag{5.9}$$

with  $N_0 \geq 0$ ,  $a \geq 0$ ,  $c > 0$  and  $b$  is an arbitrary real number. It is not difficult to see that  $a = \delta$ ,  $b = 0$  and  $c = 2$  is the squared Bessel process. A CIR process can be represented in terms of the squared Bessel process as follows

$$N_t = e^{-bt} Z \left( \frac{c^2}{4b} (e^{bt} - 1) \right),
 \tag{5.10}$$

where  $Z_t$  denotes the squared Bessel process of dimension  $\delta = \frac{4a}{c^2}$ . The above equation can be checked by using Ito’s formula for deterministic time transformation of a squared Bessel process. Using the above connection one can easily classify the properties of the origin as a boundary for the process; we just need to consider  $\frac{4a}{c^2}$  as the dimension of the corresponding Bessel process. To make the connection with the radial harmonic oscillator let’s first map the CIR process to the square root of it  $M_t = \sqrt{|N_t|}$  by using the Ito’s formula as follows

$$dM_t = \left( \left( \frac{a}{2} - \frac{c^2}{8} \right) \frac{1}{M} - \frac{b}{2} M \right) dt + \frac{c}{2} dB_t.
 \tag{5.11}$$

One can easily map the above process to the radial harmonic oscillator by the following Hamiltonian

$$H = \frac{1}{2} \left[ p^2 + \frac{\omega^2}{4} x^2 + \frac{k}{x^2} + e \right],
 \tag{5.12}$$

where  $\omega^2 = b^2$ ,  $k = \frac{1}{2}(a - \frac{c^2}{4})(\frac{a}{2} - \frac{c^2}{8} - 1)$  and  $e = -\frac{b}{2}(a - \frac{c^2}{4}) - b$ . The above Hamiltonian is just the radial part of the harmonic oscillator in  $\delta$  dimension. The above calculation shows that by deterministic time change one can go from scale invariant Bessel process to non-scale invariant process with stationary solution. This is a hint to believe that it should be possible to move from a conformal quantum mechanics to a radial harmonic oscillator by time translation. This is in fact natural and was done long time ago in [12]. The strategy is as follows: Firstly one could write  $SO(2, 1)$  generators in a more familiar form by defining new generators as follows

$$\begin{aligned} S &= \frac{1}{2} \left( \frac{1}{a} K - aH \right), \\ R &= \frac{1}{2} \left( \frac{1}{a} K + aH \right), \end{aligned} \tag{5.13}$$

where  $a$  is a constant and the commutators are as follows

$$[D, R] = iS, \quad [S, R] = -iD, \quad [S, D] = -iD. \tag{5.14}$$

The important point is that the operators  $D$  and  $S$  correspond to hyperbolic non-compact transformations and  $R$  is the generator corresponding to a compact rotation. Since all of the above operators are the invariants of the action one could define a generic operator as

$$G = uH + vD + wK, \tag{5.15}$$

as a constant of the motion. Of course it will correspond to compact rotation in three dimensions if we consider  $\Delta = v^2 - 4uw < 0$ . This will be important to get a theory with reasonable time evolution. From now on we will just consider this case. The simplest example of this kind of operators is  $R$  with  $\Delta = -1$ . The action of the operator  $G$  on a wave function is as follows

$$G|\Psi(t)\rangle = i(u + vt + wt^2) \frac{d}{dt} |\Psi(t)\rangle \tag{5.16}$$

which by time transformation could be written as

$$G|\Psi(\tau)\rangle = i \frac{d}{d\tau} |\Psi(\tau)\rangle, \tag{5.17}$$

$$\tau = \frac{4w}{\sqrt{-\Delta}} \left\{ \arctan\left(\frac{2wt + v}{-\Delta}\right) - \arctan\left(\frac{v}{-\Delta}\right) \right\}. \tag{5.18}$$

In the new parametrization one could think about  $G$  as the new time translation and thus the new Hamiltonian. After some algebra one could write the following Hamiltonian

$$\tilde{H} = \frac{1}{2} \left[ p^2 - \frac{\Delta}{4} x^2 + \frac{g}{x^2} \right], \tag{5.19}$$

as the most general possible Hamiltonian that could be extracted from the  $SO(2, 1)$  group with applicable time translation. To compare with the CIR model one could write  $\Delta = -b^2$  and  $g = k$  which is an indication to believe that the corresponding time translation in a Bessel process is related to a compact rotation in a three dimensional space with metric  $(-1, -1, 1)$ . The generators of rotations in the planes  $xy$ ,  $yz$  and  $zx$  are  $R$ ,  $D$  and  $S$  respectively. Different aspects of the self adjoint extension of the above Hamiltonian were



discussed in [38] and references therein and they are quite similar to the case without harmonic potential.

### 5.4 Morse Potential

In this subsection for the sake of completeness we want to discuss briefly another related physical model, Schrödinger equation with Morse potential [39]. This potential is exactly solvable and has many applications in molecular physics [39–41]. This system is related to the quantum mechanics of the radial Harmonic oscillator. One can find the Morse potential by just making the variable change  $u = -2 \ln x$  in the Schrödinger equation,  $H\psi(x) = E\psi(x)$  with Hamiltonian (5.12). Then we will have the following Schrödinger equation with the corresponding Hamiltonian  $H^m\phi(u) = E^m\phi(u)$  where

$$H^m = \frac{1}{2}p^2 + \frac{\omega^2}{32}e^{-2u} - \frac{E - e/2}{4}e^{-u}, \quad E^m = -\frac{1}{4}\left(\frac{1}{8} + \frac{k}{2}\right). \tag{5.20}$$

The above equality means that the Morse potential is just the canonical transformation, i.e.  $u = -2 \ln x$  and  $p_u = -\frac{1}{2}xp_x$ , of the radial Harmonic oscillator and so the dynamical symmetry group of the system is still  $SO(2, 1)$  with the transformed coordinates [42–44]. The other important issue is that in this case the domain of the quantum particle is all of the real line and so it is not necessary to worry about the boundary condition at the origin. It is also easy to extract the corresponding stochastic process by using the Hamiltonian as follows

$$dU_t = (fe^{-U_t} + l)dt + dB_t, \tag{5.21}$$

where  $f = -\frac{\omega}{4}$  and  $l = \frac{1}{2\omega}(E - e/2) + \frac{1}{2}$ . It is also possible to extract the above equation by using (5.11) and the Ito’s formula plus time re-parametrization. The connection of the Morse potential to functionals of the Brownian motion was discussed before in [45]. It is easy to see by the Feynman-Kac formula that the Kernel of the Morse Potential is equal to the following expectation in a stochastic process

$$E\left[\exp\left(\lambda ka_t - \frac{1}{2}\lambda^2 A_t\right) \middle| B_t = y\right]; \tag{5.22}$$

with  $\lambda = \frac{\omega}{4}$  and  $k = \frac{E-e/2}{\omega}$  and

$$a_t = \int_0^t \exp(B_s)ds; \quad A_t = \int_0^t \exp(2B_s)ds. \tag{5.23}$$

In [45] the connection of the above process to the Maass Laplacian [46] were also discussed extensively.

## 6 Conclusion

In this paper we explained many aspects of the connection between the Bessel process and its possible generalizations on the one side, and the conformal quantum mechanics and its generalizations on the other side. The Bessel process as the path integral interpretation of conformal quantum mechanics has conformal symmetry before considering the non-Feller boundary conditions of the process which correspond to the self adjoint extension of the

corresponding quantum mechanics. These boundary conditions could be Feller type if we consider more generalized stochastic equations. We also discussed some generalizations of the Bessel process that have interesting well-known quantum system counterparts. These generalizations are based on the connection between the Green’s function of the quantum particle and transition density of the Bessel process. Of course there are also many other systems but we focused on those that have conformal symmetry as the dynamical symmetry of system. This work could be extended in many directions including a rigorous study of the stochastic processes that correspond to the self adjoint extension of the singular quantum mechanics in the finite domain. This could be done by using the definition of local time. Investigating time inversion properties of the generalized Bessel processes in the wide sense can be useful in classification of time invertible processes.

Another interesting study could be the study of the process and quantum mechanics as a system with supersymmetry; in this case we will have superconformal symmetry as the symmetry of the quantum mechanics.

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### Appendix A: Self Adjoint Extension of the Hamiltonian

In this appendix we will summarize Von Neumann-Weyl method of self adjoint extension for the Hamiltonian operators [22, 47].

Consider a Hilbert space  $\mathcal{H}$  then an operator  $(A, \mathcal{D}(A))$  defined on  $\mathcal{H}$  is said to be densely defined if the subset  $\mathcal{D}(A)$  is dense in  $\mathcal{H}$ , i.e., that for any  $\psi \in \mathcal{H}$  one can find in  $\mathcal{D}(A)$  a sequence  $\phi_n$  which converges in norm to  $\psi$ , in other words we should have  $\int_0^\infty |\psi - \phi_n|^2 dx < \epsilon$  for arbitrary positive  $\epsilon$ .

The adjoint operator of an operator  $H$  with dense domain  $\mathcal{D}(H^\dagger)$  is  $H^\dagger$ . The domain  $\mathcal{D}(H^\dagger)$  is the space of functions  $\psi$  such that the linear form  $\phi \rightarrow (\psi, H\phi)$  is continuous for the norm of  $\mathcal{H}$  which guaranties the existence of a  $\psi^\dagger \in \mathcal{H}$  such that

$$(\psi, H\phi) = (\psi^\dagger, \phi). \tag{A.1}$$

Then one may define  $H^\dagger\psi = \psi^\dagger$ . An operator  $(H, \mathcal{D}(H))$  is said to be symmetric or Hermitian if for  $\phi, \psi \in \mathcal{D}(H)$  we have  $(\phi, H\psi) = (H\phi, \psi)$ . The operator  $H$  with the dense domain  $\mathcal{D}(H)$  is said to be self-adjoint if  $\mathcal{D}(H^\dagger) = \mathcal{D}(H)$  and  $H^\dagger = H$ .

Definition of the deficiency subspaces  $\mathcal{K}_\pm$  are by

$$\mathcal{K}_\pm = \{\psi \in \mathcal{D}(H^\dagger), H^\dagger\psi = \pm i\psi\}, \tag{A.2}$$

with dimensions  $n_\pm$  which are called the deficiency indices of the operator  $H$  and will be denoted by the ordered pair  $(n_+, n_-)$ . The following theorem, discovered by Weyl and generalized by Von Neumann, is the most important result of this appendix.

**Theorem** For an operator  $H$  with deficiency indices  $(n_+, n_-)$  there are three possibilities:

1. If  $n_+ = n_- = 0$ , then  $H$  is self-adjoint.
2. If  $n_+ = n_- = n$ , then  $H$  has infinitely many selfadjoint extensions, parameterized by a unitary  $n \times n$  matrix with  $n^2$  real parameters.
3. If  $n_+ \neq n_-$ , then  $H$  has no self-adjoint extension.

A relevant example for the above theorem is the Bessel operator discussed in the paper. The case on the half line was discussed extensively in Sect. 2 and we will not discuss it again but the case in the finite interval  $[0, L]$  is more complicated and needs to be discussed separately. In this case for  $\nu \geq 1$  only one solution is possible and the deficiency indices are  $(1, 1)$ , while for  $0 \leq \nu < 1$  both solutions are acceptable and the deficiency indices are  $(2, 2)$ . This is quite natural because for the Bessel operator in the finite interval we have another boundary which is like infinite well, so we can expect that the boundary condition is like the boundary condition of  $\delta = 3$  case for  $\nu \geq 1$ . For the case  $0 \leq \nu < 1$  the situation is more subtle because it is also possible to have some interacting boundary conditions. In almost all applications the conditions are separated and so one can look at the non-singular boundary, i.e.  $L$ , as an infinite well with one parameter extension as the  $\delta = 3$  case. The boundary condition at the origin is just as before. The interesting point is, since for a differential operator of order  $n$  with deficiency indices  $(n, n)$ , all of its self-adjoint extensions have discrete spectrums one could argue that for this case all of the energy levels are discrete. In the other words for the particle in the finite sphere with origin removed the spectrum of energy is completely discrete for  $0 < \delta < 4$ .

### Appendix B: Boundary Conditions for the Stochastic Equations

In this appendix we would like to summarize Feller’s classification of possible boundary conditions for the one dimensional stochastic equation [48, 49]. Consider the following equation as our stochastic equation

$$dx_t = \mu(x_t)dt + \sigma(x_t)dB_t. \tag{B.1}$$

To classify the possible boundary conditions we need to define the following two functions as the scale function  $s(x)$  and speed measure  $m(x)$  as follows

$$s(x) := \exp\left(-\int^x \frac{2\mu(x')}{\sigma^2(x')}dx'\right), \quad m(x) := \frac{2}{\sigma^2(x)s(x)}. \tag{B.2}$$

Using the above functions one can define the following four different functions for diffusion in the interval with endpoints  $l$  and  $r$

$$S[x, y] = \int_x^y s(z)dz, \quad S(l, y) = \lim_{x \rightarrow l^+} S[x, y], \quad S[x, r] = \lim_{x \rightarrow l^-} S[x, y], \tag{B.3}$$

$$M(c, d) = \int_c^d m(x)dx, \quad M(l, y) = \lim_{x \rightarrow l^+} M[x, y], \quad M[x, r] = \lim_{x \rightarrow l^-} M[x, y], \tag{B.4}$$

$$\sum(l) = \int_l^x S(l, y)m(z)dz, \quad \sum(r) = \int_x^r S[z, r)m(z)dz, \tag{B.5}$$

$$N(l) = \int_l^x S[z, x)m(z)dz, \quad N(r) = \int_x^r S[x, z)m(z)dz. \tag{B.6}$$

The boundary classification depends on the behavior of the above functions and one can put the possible boundary conditions in one of the following four types for the endpoint  $e$ :

1. regular if  $\sum(e)$  and  $N(e)$  be finite,
2. exit if  $\sum(e)$  be finite and  $N(e)$  be infinity,
3. entrance if  $\sum(e)$  be infinite and the  $N$  be finite,
4. natural if  $\sum(e)$  and  $N(e)$  be infinity.

For entrance, exit and natural, no boundary conditions are needed but for regular boundary, the conditional probability is not unique and dependent on the boundary conditions. An exit boundary can be reached from the interior point of the domain with positive probability however it is not possible to start the process from the exit boundary. An entrance boundary cannot be reached from the interior point of the domain but it is possible to start the process from the entrance boundary. A natural boundary cannot be reached in finite time from the interior point of the domain and it is impossible to start the process from the natural boundary. A regular boundary is accessible and could be reflecting if  $m(e) = 0$  and sticky if  $m(e) > 0$ . For reflecting boundary point the process spends no time in that point but for sticky boundary point the process spends a positive amount of time at a sticky point. For example  $m(e) = \infty$  is called a killing boundary condition.

For the Bessel process the scale function and speed measure are given by

$$s(x) = \begin{cases} \frac{1}{\nu} x^\nu & \text{if } \nu \neq 0, \\ \ln x & \text{if } \nu = 0, \end{cases} \quad m(x) = \frac{1}{2} x^{-\nu}. \quad (\text{B.7})$$

Different boundary possibilities for the Bessel process were discussed already in the paper.

## References

1. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. Springer, New York (1999)
2. Borodin, A.N., Salminen, P.: Handbook of Brownian Motion—Facts and Formulae, 2nd edn. Birkhauser, Basel-Boston-Berlin (2002)
3. Schramm, O.: Isr. J. Math. **128**, 221 (2000)
4. Cox, J.C., Ingersoll, J.E., Ross, S.A.: Econometrica **53**, 385–407 (1985)
5. Nelson, E.: Phys. Rev. **150**, 1079–1085 (1966)
6. Calogero, F.: J. Math. Phys. **10**, 2191 (1969)
7. Calogero, F.: J. Math. Phys. **10**, 2197 (1969)
8. Calogero, F.: J. Math. Phys. **12**, 419 (1971)
9. Olshanetsky, M.A., Perelomov, A.M.: Phys. Rep. **71**, 313–400 (1981)
10. Basu-Mallick, B., Ghosh, P.K., Gupta, K.S.: Phys. Lett. A **311**, 87 (2003)
11. Basu-Mallick, B., Ghosh, P.K., Gupta, K.S.: Nucl. Phys. B **659**, 437 (2003)
12. de Alfaro, V., Fubini, S., Furlan, G.: Il Nuovo Cimento A **34**, 569–612 (1976)
13. Callan, C.G., Coleman, S., Jackiw, R.: A new improved energy-momentum tensor. Ann. Phys. (NY) **59**, 42 (1970)
14. Jackiw, R.: Introducing scale symmetry. Phys. Today **25**, 23 (1972)
15. Claus, P., Derix, M., Kallosh, R., Kumar, J., Townsend, P.K., Van Proeyen, A.: Phys. Rev. Lett. **81**, 4553 (1998)
16. Gibbons, G.W., Townsend, P.K.: Phys. Lett. B **454**, 187 (1999)
17. Camblong, H.E., Epele, L.N., Fanchiotti, H., Garcia Canal, C.A.: Phys. Rev. Lett. **87**, 220402 (2001). arXiv:[hep-th/0106144](https://arxiv.org/abs/hep-th/0106144)
18. Lawler, G.F.: Commun. Math. Phys. **86**, 539–554 (1982)
19. Göing-Jaeschke, A., Yor, M.: A survey and some generalizations of Besel processes. ETH Zürich
20. Case, K.M.: Phys. Rev. **80**, 797–806 (1950)
21. Frank, W.M., Land, D.J., Spector, R.M.: Rev. Mod. Phys. **43**, 36–98 (1971)
22. Hutson, V.C.L., Pym, J.S.: Application of Functional Analysis and Operator Theory. Academic Press, San Diego (1980)
23. Basu-Mallick, B., Gupta, K.S.: Phys. Lett. A **292**, 36–42 (2001)
24. Essin, A.M., Griffiths, D.J.: Am. J. Phys. **74**, 109–117 (2006)
25. Basu-Mallick, B., Ghosh, P.K., Gupta, K.S.: Nucl. Phys. B **659**, 437–457 (2003). arXiv:[hep-th/0207040](https://arxiv.org/abs/hep-th/0207040)

26. Ananos, G.N.J., Camblong, H.E., Ordonez, C.R.: Phys. Rev. D **68**, 025006 (2003). arXiv:[hep-th/0302197](#)
27. Camblong, H.E., Ordonez, C.R.: Phys. Rev. D **68**, 125013 (2003). arXiv:[hep-th/0303166](#)
28. Jackiw, R.: Delta-function potentials in two- and three-dimensional quantum mechanics. In Ali, A., Hoodbhoy, P. (eds.) M.A.B. Beg Memorial Volume. World Scientific, Singapore (1991)
29. Friedman, C.: J. Funct. Anal. **10**, 346 (1972)
30. Clark, T.E., Menikoff, R., Sharp, D.H.: Phys. Rev. D **22**, 3012–3016 (1980)
31. Farhi, E., Gutmann, S.C.: Int. J. Mod. Phys. A **5**, 3029 (1990)
32. Carreau, M., Farhi, E., Gutmann, S., Mende, P.F.: Ann. Phys. (NY) **204**, 186–207 (1990)
33. Linetsky, V.: J. Appl. Probab. **41**(2), 327–344 (2004)
34. Watanabe, S., Wahrsch, Z.: Verw. Geb. **31**, 115–124 (1975)
35. Rogers, L.C.G., Pitman, J.W.: Ann. Probab. **9**, 573–582 (1981)
36. Lawi, S.: arXiv:[math-PR/0506013](#)
37. Heston, S.L.: Rev. Financ. Stud. **6**(2), 327–343 (1993)
38. Falomir, H., Pisani, P.A.G., Wipf, A.: J. Phys. A **35**, 5427–5444 (2002)
39. Morse, P.M.: Phys. Rev. **34**, 57 (1929)
40. Pauling, L., Wilson, E.B.: Introduction to Quantum Mechanics with Applications to Chemistry. Dover, New York (1985)
41. Jensen, P.: Mol. Phys. **98**, 1253 (2000)
42. Cooper, I.L., Gupta, R.K.: Phys. Rev. A **52**, 941 (1995)
43. Gerry, C.C.: Phys. Rev. A **33**, 2207 (1986)
44. Bessis, N., Bessis, G.: Phys. Rev. A **50**, 4506 (1994)
45. Ikedaa, N., Matsumoto, H.: J. Funct. Anal. **163**, 63–110 (1999)
46. Fay, J.D.: J. Reine Angew. Math. **293**, 143 (1977)
47. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Academic Press, San Diego (1975), vol. I and II
48. Feller, W.: Ann. Math. **55**, 468–519 (1952)
49. Ito, K., McKean Jr., H.P.: Diffusion Processes and their Sample Paths. Springer, Berlin (1965)